

# Fermionic Casimir densities in anti-de Sitter spacetime

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## Abstract

The fermionic condensate and vacuum expectation value of the energy-momentum tensor, for a massive fermionic field on the background of anti-de Sitter spacetime, in the geometry of two parallel boundaries with bag boundary conditions, are investigated. Vacuum expectation values, expressed as series involving the eigenvalues of the radial quantum number, are neatly decomposed into boundary-free, single-boundary-induced, and second-boundary-induced parts, with the help of the generalized Abel-Plana summation formula. In this way, the renormalization procedure is very conveniently reduced to the one corresponding to boundary-free AdS spacetime. The boundary-induced contributions to the fermionic condensate and to the vacuum expectation value of the energy density are proven to be everywhere negative. The vacuum expectation values are exponentially suppressed at distances from the boundaries much larger than the curvature radius of the AdS space. Near the boundaries, effects related with the curvature of the background spacetime are shown to be subdominant and, to leading order, all known results for boundaries in the Minkowski bulk are recovered. Zeta function techniques are successfully used for the evaluation of the total vacuum energy in the region between the boundaries. It is proven that the resulting interaction forces between them are attractive and that, for large separations, they also decay exponentially. Finally, our results are extended and explicitly translated to fermionic Casimir densities in braneworld scenarios of Randall-Sundrum type.

## 1 Introduction

Anti-de Sitter (AdS) spacetime is among the most popular background geometries in quantum field theory. Much of the earlier interest in this geometry was motivated by questions of principal nature, mainly related with the quantization of fields on curved backgrounds. The AdS spacetime has maximal symmetry and, because of this, numerous physical problems can be exactly solved on this background.

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The presence of both regular and irregular modes and the possibility of getting an interesting causal structure lead to several new and remarkable phenomena. Further interest in this subject arose from the discovery that the AdS spacetime generically arises as a ground state in extended supergravity and string theories, what is again potentially most important.

In recent developments of the topic, the AdS geometry is an arena for two classes of models. The first is the AdS/CFT correspondence (for a review see [1]), which represents a realization of the holographic principle and relates string theories or supergravity in the AdS bulk with a conformal field theory living on its boundary. This correspondence has many interesting consequences and provides a powerful tool for the investigation of physical effects in gauge theories. The second class of models with the AdS spacetime as background geometry is a realization of the braneworld scenario with large extra dimensions and provides a solution to the hierarchy problem which arises between the gravitational and electroweak mass scales (for reviews on braneworld gravity and cosmology see [2]). Here the small coupling of 4-dimensional gravity is generated by the large physical volume of extra dimensions. Braneworlds naturally appear in the string/M theory context and provide a novel set up for the discussion of phenomenological and cosmological issues related with extra dimensions.

In the present paper, as an example of an exactly solvable physical problem in AdS spacetime, we will consider the Casimir effect (for reviews see [3]) for a fermionic field obeying bag boundary conditions on two parallel plates. The explicit dependence of the characteristics of the vacuum on the geometry of the background spacetime is among the most interesting topics in the investigation of the Casimir effect. As usual, all relevant information is encoded in the spectrum of the vacuum fluctuations and, not surprisingly, analytic solutions can only be found, in general, for highly symmetric geometries. Specifically, the Casimir effect for a massive scalar field with general curvature coupling parameter in the geometry of flat and spherical boundaries on the background of de Sitter spacetime has been investigated recently in [4] and [5], respectively.

Investigations of the Casimir effect in AdS spacetime have already attracted a great deal of attention, motivated by Randall-Sundrum type braneworld scenarios [6]. In these models the background solution consists of two parallel flat branes embedded in a 5-dimensional AdS bulk. The fifth coordinate is compactified on  $S^1/Z^2$ , and the branes are on two fixed points. The fields which propagate in the bulk will give Casimir-type contributions to the vacuum energy and, as a result, to the vacuum forces acting on the branes. The Casimir effect provides in this context a natural mechanism for stabilizing the radion field, as required for a complete solution of the hierarchy problem. In addition, the Casimir energy gives a contribution to both the brane and the bulk cosmological constants and, hence, it has to be taken into account in any self-consistent formulation of the braneworld dynamics. The Casimir energy and the corresponding forces for two parallel branes in AdS spacetime have been evaluated in Refs. [7, 8], both for scalar and fermionic fields, by using either dimensional or zeta function regularization methods. Local Casimir densities were considered in Refs. [9, 10, 11]. The Casimir effect in higher-dimensional generalizations of the AdS spacetime with compact internal spaces has been investigated in [12] while the Casimir energy for a massless fermionic field with generalized bag boundary conditions in 3-dimensional AdS spacetime was discussed in [13], for the geometry where one of the boundaries coincides with the AdS boundary.

In braneworld models, two distinct types of boundary conditions arise for Dirac fermion fields, corresponding respectively to even and odd fields (untwisted and twisted boundary conditions) [14]. In the present paper, we investigate one-loop quantum effects for a fermionic field in AdS spacetime, induced by two parallel boundaries with bag boundary condition. Although this condition is different from those appearing in braneworld scenarios, we will here show in detail how, from our formulas, the corresponding results for braneworlds are readily obtained. The important quantities that characterize the local properties of the fermionic vacuum are the fermionic condensate (FC) and the vacuum expectation value (VEV) of the energy-momentum tensor. For the investigation of these quantities we use the direct mode summation approach, which requires the knowledge of a complete set of mode functions for the fermionic field obeying the boundary conditions.

In the next section, we describe the geometry of the problem and construct the corresponding mode functions. By making use of them, in Sect. 3 we evaluate the FC. Applying the generalized Abel-Plana formula, the FC is decomposed into three parts: a boundary-free contribution, a single-boundary-induced one, and a second-boundary-induced one. The behavior of the VEVs in asymptotic regions of the parameters is discussed. In Sect. 4 we present similar considerations for the VEV of the energy-momentum tensor. Casimir forces acting on the boundaries and the corresponding Casimir energy are investigated in Sect. 5. The generalization of the formulas obtained to the important case of a bulk fermionic field in the Randall-Sundrum braneworld model is discussed. Finally, our main results are summarized in Sect. 6.

## 2 Fermionic mode functions

We consider a quantum fermionic field,  $\psi$ , in  $(D+1)$ -dimensional anti-de Sitter spacetime,  $AdS_{D+1}$ . In Poincaré coordinates, the corresponding line element reads

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{-2y/a} \eta_{ik} dx^i dx^k - dy^2, \quad (2.1)$$

with  $\eta_{ik} = \text{diag}(1, -1, \dots, -1)$ ,  $i, k = 0, \dots, D-1$ , being the Minkowskian metric tensor for a  $D$ -dimensional spacetime. In (2.1),  $a$  is the AdS curvature radius which is related to the Ricci scalar by  $R = -D(D+1)/a^2$ . In addition to the radial coordinate  $y$  we will use the conformal coordinate  $z$ , defined as  $z = ae^{y/a}$ . With this coordinate, the AdS line element is written in conformally-flat form:

$$ds^2 = (a/z)^2 \left( \eta_{ik} dx^i dx^k - dz^2 \right). \quad (2.2)$$

The hypersurfaces  $z = 0$  and  $z = \infty$  correspond to the AdS boundary and to the horizon, respectively.

The dynamics of a fermionic field in curved spacetime are governed by the covariant Dirac equation

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0, \quad \nabla_\mu = \partial_\mu + \Gamma_\mu, \quad (2.3)$$

where  $\Gamma_\mu$  is the spin connection. The Dirac matrices  $\gamma^\mu$  are expressed in terms of the flat space-time gamma matrices  $\gamma^{(a)}$ , as  $\gamma^\mu = e_{(a)}^\mu \gamma^{(a)}$ , with  $e_{(a)}^\mu$  being the tetrad fields obeying the relation  $e_{(a)}^\mu e_{(b)}^\nu \eta^{ab} = g^{\mu\nu}$ . For the spin connection, one has

$$\Gamma_\mu = \frac{1}{4} \gamma^{(a)} \gamma^{(b)} e_{(a)}^\nu e_{(b)\nu;\mu}, \quad (2.4)$$

where the semicolon means covariant derivative of vector fields. For the geometry under consideration the tetrads can be taken in the form  $e_{(b)}^\mu = \delta_b^\mu z/a$ . With this choice, the spin connection has the following components (no summation over  $l$ )

$$\Gamma_D = 0, \quad \Gamma_l = \frac{\eta_l}{2z} \gamma^{(D)} \gamma^{(l)}, \quad l = 0, \dots, D-1. \quad (2.5)$$

For the combination appearing in the Dirac equation, we have  $\gamma^\mu \Gamma_\mu = -D\gamma^{(D)}/(2a)$ .

We are interested in the change of the properties of the fermionic vacuum induced by the presence of the two boundaries, which are located at  $z = z_1$  and  $z = z_2$ ,  $z_1 < z_2$ . The corresponding values of the physical radial coordinate,  $y$ , will be denoted by  $y_1$  and  $y_2$ :  $z_j = ae^{y_j/a}$ ,  $j = 1, 2$ . We will assume that, on the boundaries, the field obeys the bag boundary condition, namely

$$(1 + i\gamma^\mu n_\mu^{(j)})\psi = 0, \quad z = z_j, \quad (2.6)$$

$j = 1, 2$ , with  $n_\mu^{(j)}$  being normal to the boundaries,  $n_\mu = (-1)^j \delta_\mu^D a/z$ . From these conditions it follows that the normal component of the fermion current vanishes at the boundaries.

In  $(D+1)$ -dimensional flat spacetime the Dirac matrices are  $N \times N$  matrices, with  $N = 2^{[(D+1)/2]}$  (the square brackets mean the integer part of the enclosed expression). In the discussion below we will assume the following representation for these matrices:

$$\gamma^{(0)} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^{(a)} = i \begin{pmatrix} -\sigma_a & 0 \\ 0 & \sigma_a \end{pmatrix}, \quad (2.7)$$

with  $a = 1, \dots, D$  and  $\sigma_a \sigma_b + \sigma_b \sigma_a = 2\delta_{ab}$ . The last relation directly follows from the anticommutation relations for the Dirac matrices. The matrices (2.7) are related to the gamma matrices in the standard Dirac representation,  $\gamma_{(D)}^{(a)}$ , by  $\gamma^{(a)} = iB\gamma_{(D)}^{(a)}$ , where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.8)$$

As we will see below, with the representation (2.7) the equations for the components of the fermionic field are conveniently separated. This fact is to be remarked, since it allows for a complete calculation.

Among the most important characteristics of the fermionic vacuum are the FC and the VEV of the energy-momentum tensor. For the evaluation of these quantities we will use the direct mode summation approach. In this approach we need a complete set of solutions to Eq. (2.3) obeying the boundary conditions (2.6). For the positive-energy solutions the dependence of the mode functions on the time and on the coordinates parallel to the boundaries, denoted as  $\mathbf{x} = (x^1, \dots, x^{D-1})$ , can be expressed in the form  $e^{i\mathbf{k}\mathbf{x} - i\omega t}$ , where  $\mathbf{k} = (k_1, \dots, k_{D-1})$ ,  $\mathbf{k}\mathbf{x} = k_l x^l$ , and the summation runs over  $l = 1, \dots, D-1$ .

Decomposing the field into upper and lower components,

$$\psi = \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix} e^{i\mathbf{k}\mathbf{x} - i\omega t}, \quad (2.9)$$

the initial Dirac equation is reduced to the set

$$\left[ \sigma_D \left( \partial_z - \frac{D}{2z} \right) + ik_l \sigma_l \mp \frac{ma}{z} \right] \psi_{\pm} - i\omega \psi_{\mp} = 0, \quad (2.10)$$

from where we can obtain separate equations for the upper and lower components:

$$(z^2 \partial_z^2 - Dz \partial_z + \lambda^2 z^2 + D^2/4 + D/2 - m^2 a^2 \pm \sigma_D ma) \psi_{\pm} = 0, \quad (2.11)$$

being  $\lambda^2 = \omega^2 - k^2$ . With the substitution

$$\psi_{\pm}(z) = z^{(D+1)/2} \chi_{\pm}(z), \quad (2.12)$$

Eq. (2.11) is reduced to the Bessel equation:

$$(z^2 \partial_z^2 + z \partial_z + \lambda^2 z^2 \pm \sigma_D ma - m^2 a^2 - 1/4) \chi_{\pm} = 0. \quad (2.13)$$

Taking the  $\sigma_D$  matrix in the form

$$\sigma_D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.14)$$

we further decompose  $\chi_{\pm}(z)$  into upper and lower components,

$$\chi_{\pm}(z) = \begin{pmatrix} \varphi_{\pm\uparrow}(z) \\ \varphi_{\pm\downarrow}(z) \end{pmatrix}. \quad (2.15)$$

The solutions for these components directly follow from (2.11), and are

$$\begin{aligned}\varphi_{\pm\uparrow} &= C_{\pm\uparrow}^{(J)} J_{ma\mp 1/2}(\lambda z) + C_{\pm\uparrow}^{(Y)} Y_{ma\mp 1/2}(\lambda z), \\ \varphi_{\pm\downarrow} &= C_{\pm\downarrow}^{(J)} J_{ma\pm 1/2}(\lambda z) + C_{\pm\downarrow}^{(Y)} Y_{ma\pm 1/2}(\lambda z).\end{aligned}\quad (2.16)$$

where  $J_\nu(x)$  and  $Y_\nu(x)$  are Bessel and Neumann functions, respectively. The relation between the coefficients in these linear combinations are obtained by using Eq. (2.10). In order to find them we note that, from the anticommutation relations for the matrices  $\sigma_D$  and  $\sigma_l$ ,  $l = 1, \dots, D-1$ , and from (2.14), it readily follows that the matrices  $\sigma_l$  have the form

$$\sigma_l = \begin{pmatrix} 0 & b_l \\ c_l & 0 \end{pmatrix},$$

with  $b_l c_l = c_l b_l = 1$  and  $b_l c_k = -b_k c_l$ ,  $c_l b_k = -c_k b_l$ , for  $l \neq k$ . Using these relations, from the equation (2.10) with the upper sign, we find

$$\begin{aligned}\omega C_{-\uparrow}^{(J)} &= k_l b_l C_{+\downarrow}^{(J)} + i\lambda C_{+\uparrow}^{(J)}, \quad \omega C_{-\uparrow}^{(Y)} = k_l b_l C_{+\downarrow}^{(Y)} + i\lambda C_{+\uparrow}^{(Y)}, \\ \omega C_{-\downarrow}^{(J)} &= k_l c_l C_{+\uparrow}^{(J)} + i\lambda C_{+\downarrow}^{(J)}, \quad \omega C_{-\downarrow}^{(Y)} = k_l c_l C_{+\uparrow}^{(Y)} + i\lambda C_{+\downarrow}^{(Y)}.\end{aligned}$$

As a result, the solution of the Dirac equation can be written in the form

$$\psi = \frac{z^{(D+1)/2}}{\omega} e^{i\mathbf{k}\mathbf{x} - i\omega t} \begin{pmatrix} \omega Z_{\uparrow, \nu-1}(\lambda z) \\ \omega Z_{\downarrow, \nu}(\lambda z) \\ k_l b_l Z_{\downarrow, \nu}(\lambda z) + i\lambda Z_{\uparrow, \nu}(\lambda z) \\ k_l c_l Z_{\uparrow, \nu-1}(\lambda z) + i\lambda Z_{\downarrow, \nu-1}(\lambda z) \end{pmatrix}, \quad (2.17)$$

with the notations

$$\nu = ma + 1/2, \quad (2.18)$$

and

$$\begin{aligned}Z_{\uparrow, \mu}(\lambda z) &= C_{+\uparrow}^{(J)} J_\mu(\lambda z) + C_{+\uparrow}^{(Y)} Y_\mu(\lambda z), \\ Z_{\downarrow, \mu}(\lambda z) &= C_{+\downarrow}^{(J)} J_\mu(\lambda z) + C_{+\downarrow}^{(Y)} Y_\mu(\lambda z).\end{aligned}\quad (2.19)$$

We now must impose the boundary conditions (2.6). From the BC at  $z = z_1$ , we have

$$\frac{C_{+\uparrow}^{(Y)}}{C_{+\uparrow}^{(J)}} = \frac{C_{+\downarrow}^{(Y)}}{C_{+\downarrow}^{(J)}} = -\frac{J_\nu(\lambda z_1)}{Y_\nu(\lambda z_1)}, \quad (2.20)$$

and hence

$$\frac{Z_{\uparrow, \mu}(\lambda z)}{C_{+\uparrow}^{(J)}} = \frac{Z_{\downarrow, \mu}(\lambda z)}{C_{+\downarrow}^{(J)}} = Z_\mu(\lambda z_1, \lambda z). \quad (2.21)$$

Here and in what follows we use the notation

$$Z_\mu(x, y) = J_\mu(y) - \frac{J_\nu(x)}{Y_\nu(x)} Y_\mu(y). \quad (2.22)$$

In order to write the solution of the Dirac equation in a more compact form we introduce the notations

$$C_+^{(J)} = \begin{pmatrix} C_{+\uparrow}^{(J)} \\ C_{+\downarrow}^{(J)} \end{pmatrix} \quad (2.23)$$

and

$$\widehat{Z}_{\pm}(x, y) = \begin{pmatrix} Z_{ma\pm 1/2}(x, y) & 0 \\ 0 & Z_{ma\mp 1/2}(x, y) \end{pmatrix}. \quad (2.24)$$

With these notations, the solution (2.17) obeying the boundary condition at  $z = z_1$  can be expressed in the form

$$\psi = z^{(D+1)/2} e^{i\mathbf{k}\mathbf{x} - i\omega t} \begin{pmatrix} \widehat{Z}_-(\lambda z_1, \lambda z) C_+^{(J)} \\ \omega^{-1} \widehat{Z}_+(\lambda z_1, \lambda z) (i\lambda + k_l \sigma_l) C_+^{(J)} \end{pmatrix}. \quad (2.25)$$

This solution corresponds to a state of the fermionic field with a given value of the momentum parallel to the boundary and with a given value of  $\lambda$ . In order to completely specify the solutions we still need an additional quantum number. This corresponds to fixing the spinor  $C_+^{(J)}$ . Here we take  $C_+^{(J)} = C_{\beta}^{(+)} w^{(\sigma)}$ , where  $C_{\beta}^{(+)}$  is a constant and  $w^{(\sigma)}$ ,  $\sigma = 1, \dots, N/2$ , are one-column matrices of  $N/2$  rows, with elements  $w_l^{(\sigma)} = \delta_{l\sigma}$ . It can be seen that with this choice, the solutions (2.25), in combination with the corresponding negative-energy solutions (see below), do form a complete set specified by the quantum numbers  $\beta = (\mathbf{k}, \lambda, \sigma)$ . Hence, the positive-energy mode functions obeying the boundary condition at  $z = z_1$  have the form

$$\psi_{\beta}^{(+)} = C_{\beta}^{(+)} z^{\frac{D+1}{2}} e^{i\mathbf{k}\mathbf{x} - i\omega t} \begin{pmatrix} \widehat{Z}_-(\lambda z_1, \lambda z) w^{(\sigma)} \\ \frac{1}{\omega} \widehat{Z}_+(\lambda z_1, \lambda z) (i\lambda + k_l \sigma_l) w^{(\sigma)} \end{pmatrix}. \quad (2.26)$$

Now we impose the boundary condition on the right boundary, located at  $z = z_2$ . From this condition it follows that the eigenvalues of the quantum number  $\lambda$  are roots of the equation

$$g_{\nu, \nu-1}(\lambda z_1, \lambda z_2) = 0, \quad (2.27)$$

where we have defined the function

$$g_{\nu, \mu}(x, y) = J_{\nu}(x) Y_{\mu}(y) - J_{\mu}(y) Y_{\nu}(x). \quad (2.28)$$

Eq. (2.27) has an infinite number of positive roots. We will denote them by  $\lambda = \lambda_n = \gamma_{\nu, n}/z_1$ ,  $n = 1, 2, \dots$

The coefficient  $C_{\beta}^{(+)}$  in (2.26) is determined from the normalization condition

$$\int d^{D-1}x \int_{z_1}^{z_2} dz \sqrt{|\gamma|} \psi_{\beta}^{(+)+} \psi_{\beta'}^{(+)} = \delta(\mathbf{k} - \mathbf{k}') \delta_{\sigma\sigma'} \delta_{nn'}, \quad (2.29)$$

where  $\gamma$  is the determinant of the spatial metric,  $|\gamma| = (a/z)^D$ . By using a standard result for the integral involving the square of the cylinder functions (see, for instance, [15]), for the normalization coefficient we find

$$|C_{\beta}^{(+)}|^2 = \frac{\pi^2 \lambda Y_{\nu}^2(\lambda z_1)}{4(2\pi)^{D-1} a^D z_1} T_{\nu}(\eta, \lambda z_1), \quad (2.30)$$

where we have introduced the notations

$$\eta = z_2/z_1, \quad (2.31)$$

and

$$T_{\nu}(\eta, x) = x \left[ \frac{J_{\nu}^2(x)}{J_{\nu-1}^2(\eta x)} - 1 \right]^{-1}. \quad (2.32)$$

This finishes the construction of the positive-energy mode functions.

The negative-energy mode functions can be obtained in a similar way. They have the form:

$$\psi_{\beta}^{(-)} = C_{\beta}^{(-)} z^{\frac{D+1}{2}} e^{i\mathbf{k}\mathbf{x} - i\omega t} \begin{pmatrix} \frac{1}{\omega} \widehat{Z}_-(\lambda z_1, \lambda z) (i\lambda - k_l \sigma_l) w^{(\sigma)} \\ \widehat{Z}_+(\lambda z_1, \lambda z) w^{(\sigma)} \end{pmatrix}, \quad (2.33)$$

where  $|C_{\beta}^{(-)}|^2$  is given by the same expression (2.30). As in the case of the positive-energy modes, for the eigenvalues of  $\lambda$  one has  $\lambda_n = \gamma_{\nu, n}/z_1$ .

### 3 Fermionic condensate

In this section we consider the FC defined as the VEV  $\langle 0 | \bar{\psi} \psi | 0 \rangle \equiv \langle \bar{\psi} \psi \rangle$ , where  $|0\rangle$  corresponds to the vacuum state and  $\bar{\psi} = \psi^\dagger \gamma^{(0)}$  is the Dirac adjoint. Note that the Dirac adjoint is defined through the flat spacetime matrix  $\gamma^{(0)}$ . In addition to describing the physical structure of the quantum field at a given point, the FC plays an important role in models of dynamical chiral symmetry breaking (see the reviews [16], for chiral symmetry breaking on curved spacetime with nontrivial topology, and [17] for recent developments).

Expanding the field operator in terms of the complete set of positive- and negative-energy mode functions  $\{\psi_\beta^{(+)}, \psi_\beta^{(-)}\}$ , and using the anticommutation relations for the annihilation and creation operators, we find the mode-sum formula for the fermionic condensate:

$$\langle \bar{\psi} \psi \rangle = \sum_\beta \bar{\psi}_\beta^{(-)} \psi_\beta^{(-)}, \quad (3.1)$$

where

$$\sum_\beta = \int d\mathbf{k} \sum_{\sigma=1}^{N/2} \sum_{n=1}^{\infty}. \quad (3.2)$$

The expression on the right-hand side (rhs) of (3.1) is divergent and, to make sense of it, some regularization procedure is needed. Here we assume that a cutoff function is present, without explicitly writing it. The special form of this function will not be important in the further discussion.

Substituting the mode functions (2.33) into (3.1) after some transformations the fermionic condensate can be expressed as

$$\langle \bar{\psi} \psi \rangle = -\frac{Na^{-D}z^{D+1}}{16(2\pi)^{D-3}z_1^2} \int d\mathbf{k} \sum_{n=1}^{\infty} \frac{\gamma_{\nu,n}^2 T_\nu(\eta, \gamma_{\nu,n})}{\sqrt{\gamma_{\nu,n}^2 + k^2 z_1^2}} g_{\nu,\nu}(\gamma_{\nu,n}, \gamma_{\nu,n}z/z_1) g_{\nu,\nu-1}(\gamma_{\nu,n}, \gamma_{\nu,n}z/z_1), \quad (3.3)$$

the function  $g_{\nu,\mu}(x, y)$  being defined by (2.28). As the roots  $\gamma_{\nu,n}$  are given implicitly, the form (3.3) for the FC is not convenient for the investigation of the effects induced by the boundaries. In addition, the terms in the series are highly oscillatory for large values of  $n$ .

A more convenient form of the mode sum for the FC is obtained by using the summation formula

$$\sum_{n=1}^{\infty} T_\nu(\eta, \gamma_{\nu,n}) h(\gamma_{\nu,n}) = \frac{2}{\pi^2} \int_0^\infty \frac{h(x) dx}{J_\nu^2(x) + Y_\nu^2(x)} + \frac{1}{2\pi} \int_0^\infty dx \Omega_\nu^{(1)}(x, \eta x) [h(ix) + h(-ix)], \quad (3.4)$$

where

$$\Omega_\nu^{(1)}(x, y) = \frac{K_{\nu-1}(y)/K_\nu(x)}{K_\nu(x)I_{\nu-1}(y) + I_\nu(x)K_{\nu-1}(y)}, \quad (3.5)$$

and  $I_\nu(x)$ ,  $K_\nu(x)$  are modified Bessel functions. This formula is derived in Ref. [18] by using the generalized Abel-Plana formula (see also [19]). The corresponding conditions on the function  $h(u)$ , analytic on the right half plane of the complex variable  $u$ , can be found in [18]. Applying expression (3.4) to the series over  $n$  in (3.3), after the integration over the angular part of  $\mathbf{k}$ , and introducing a new integration variable  $x = kz_1$ , the FC can be written as

$$\begin{aligned} \langle \bar{\psi} \psi \rangle &= \langle \bar{\psi} \psi \rangle_1 + \frac{8N(z/z_1)^{D+1} a^{-D}}{(4\pi)^{(D+1)/2} \Gamma((D-1)/2)} \int_0^\infty dx x^{D-2} \\ &\times \int_x^\infty du \frac{u^2 \Omega_\nu^{(1)}(u, \eta u)}{\sqrt{u^2 - x^2}} G_{\nu,\nu}(u, uz/z_1) G_{\nu,\nu-1}(u, uz/z_1). \end{aligned} \quad (3.6)$$

where we have introduced the notation

$$G_{\nu,\mu}(x, y) = I_\nu(x) K_\mu(y) - (-1)^{\nu-\mu} I_\mu(y) K_\nu(x). \quad (3.7)$$

The first term on the rhs of (3.6) comes from the first integral in (3.4) and it is given by

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_1 &= -\frac{N(z/z_1)^{D+1}a^{-D}}{(4\pi)^{(D-1)/2}\Gamma((D-1)/2)} \int_0^\infty dx x^{D-2} \int_0^\infty du \\ &\times \frac{u^2}{\sqrt{u^2+x^2}} \frac{g_{\nu,\nu}(u, uz/z_1)g_{\nu,\nu-1}(u, uz/z_1)}{J_\nu^2(u) + Y_\nu^2(u)}. \end{aligned} \quad (3.8)$$

We now first consider this term.

### 3.1 Condensate for the geometry of a single boundary

The second term on the rhs of (3.6) is finite for  $z < z_2$ , in the absence of the cutoff function. This term vanishes in the limit  $z_2 \rightarrow \infty$ , whereas  $\langle \bar{\psi}\psi \rangle_1$  does not depend on  $z_2$ . This allows us to interpret the part (3.8) as the FC in the region  $z > z_1$  for the geometry of a single boundary at  $z = z_1$  when the other boundary is absent. This can also be seen by direct evaluation of the FC using Eq. (3.1). The corresponding positive-energy mode functions are given by (2.26), where now the spectrum for  $\lambda$  is continuous. Consequently, in (3.1) we have  $\sum_\beta = \int d\mathbf{k} \int_0^\infty d\lambda \sum_{\sigma=1}^{N/2}$ . The normalization coefficient is determined by the condition which is obtained from (2.29) by the replacements  $\int_{z_1}^{z_2} dz \rightarrow \int_{z_1}^\infty dz$  and  $\delta_{nn'} \rightarrow \delta(\lambda - \lambda')$ . In this way, we can see that

$$|C_\beta^{(+)}|^2 = \frac{a^{-D}\lambda}{2(2\pi)^{D-1}} \left[ 1 + \frac{J_\nu^2(\lambda z_1)}{Y_\nu^2(\lambda z_1)} \right]^{-1}. \quad (3.9)$$

Similarly, the negative-energy modes have the form (2.33) with  $|C_\beta^{(-)}|^2$  given by the same expression (3.9). Substituting these mode functions into the formula (3.1), the expression (3.8) is obtained for the single boundary part.

For further transformation of such expression we use the identity

$$\frac{g_{\nu,\nu}(u, y)g_{\nu,\nu-1}(u, y)}{J_\nu^2(u) + Y_\nu^2(u)} = J_\nu(y)J_{\nu-1}(y) - \frac{1}{2} \sum_{s=1,2} \frac{J_\nu(u)}{H_\nu^{(s)}(u)} H_\nu^{(s)}(y) H_{\nu-1}^{(s)}(y), \quad (3.10)$$

the  $H_\nu^{(s)}(y)$ ,  $s = 1, 2$ , being Hankel functions. Substituting (3.10) into (3.8), in the integral over  $u$  with the second term on the rhs of (3.10) we rotate the integration contour in the complex plane  $u$  by the angle  $\pi/2$  ( $-\pi/2$ ) for the term with  $s = 1$  ( $s = 2$ ). Introducing the modified Bessel functions, the FC is then decomposed into

$$\langle \bar{\psi}\psi \rangle_1 = \langle \bar{\psi}\psi \rangle_0 + \langle \bar{\psi}\psi \rangle_1^{(b)}, \quad (3.11)$$

with

$$\langle \bar{\psi}\psi \rangle_0 = -\frac{(4\pi)^{(1-D)/2}N}{\Gamma((D-1)/2)a^D} \int_0^\infty dx x^{D-2} \int_0^\infty dy \frac{y^2 J_\nu(y)}{\sqrt{y^2+x^2}} J_{\nu-1}(y) \quad (3.12)$$

and

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_1^{(b)} &= -\frac{N(z/z_1)^{D+1}a^{-D}}{(4\pi)^{(D-1)/2}\Gamma((D-1)/2)} \int_0^\infty dx x^{D-2} \\ &\times \frac{2}{\pi} \int_x^\infty du \frac{u^2}{\sqrt{u^2-x^2}} \frac{I_\nu(u)}{K_\nu(u)} K_\nu(uz/z_1) K_{\nu-1}(uz/z_1). \end{aligned} \quad (3.13)$$

The second part (3.13) is finite for  $z > z_1$  and the cutoff function need be kept in the first part (3.12) only. Note that the latter does not depend on  $z$ . The term  $\langle \bar{\psi}\psi \rangle_1^{(b)}$  vanishes in the limit  $z \rightarrow \infty$  and, hence, the term  $\langle \bar{\psi}\psi \rangle_0$  can be interpreted as the pure AdS part of the FC when the boundaries are absent. The property that this term is uniform is just a consequence of the maximal symmetry



both of AdS spacetime and of the vacuum state we have chosen. With the representation (3.11), the renormalization of the FC outside the boundary is thus reduced to the one corresponding to AdS spacetime when the boundaries are absent.

The expression (3.13) for the boundary-induced part can be further simplified by using the integration formula

$$\int_0^\infty dx x^{D-2} \int_x^\infty du \frac{f(u)}{\sqrt{u^2 - x^2}} = \sqrt{\pi} \frac{\Gamma((D-1)/2)}{2\Gamma(D/2)} \int_0^\infty dr r^{D-2} f(r). \quad (3.14)$$

This is obtained by introducing a new integration variable  $y = \sqrt{u^2 - x^2}$ , passing to polar coordinates on the plane  $(x, y)$  and then integrating over the polar angle. Now, the expression for the FC induced in the region  $z > z_j$  by a single boundary, located at  $z = z_j$ , simply reads

$$\langle \bar{\psi} \psi \rangle_j^{(b)} = -\frac{2N(z/z_j)^{D+1}}{A_D a^D} \int_0^\infty dx x^D \frac{I_\nu(x)}{K_\nu(x)} K_\nu(xz/z_j) K_{\nu-1}(xz/z_j), \quad (3.15)$$

where we have introduced the notation

$$A_D = (4\pi)^{D/2} \Gamma(D/2). \quad (3.16)$$

Note that the boundary-induced part (3.15) is a function of the ratio  $z/z_j$  alone. By taking into account that  $z/z_j = e^{(y-y_j)/a}$ , we see that for a given distance from the boundary,  $y - y_j$ , the quantity  $\langle \bar{\psi} \psi \rangle_1^{(b)}$  does not depend on the location of the boundary. Again, this is a consequence of the maximal symmetry of the AdS bulk. It follows from (3.15) that  $\langle \bar{\psi} \psi \rangle_1^{(b)}$  is negative.

The expression (3.15) gives the boundary-induced part in the region  $z > z_j$ . In order to find the condensate induced by a single boundary in the region to the left of the boundary, we take the limit  $z_1 \rightarrow 0$  in the general expression (3.6) for the geometry with two boundaries. Introducing in (3.15), with  $j = 1$ , a new integration variable,  $y = x/z_1$ , we see that in this limit  $\langle \bar{\psi} \psi \rangle_1^{(b)}$  vanishes and  $\langle \bar{\psi} \psi \rangle_1$  reduces to  $\langle \bar{\psi} \psi \rangle_0$ . In the second term on the rhs of (3.6), passing to the new integration variables  $u' = u/z_1$  and  $x' = x/z_1$ , the limit  $z_1 \rightarrow 0$  is readily evaluated. As a result, from (3.6) we obtain the FC in the region  $z < z_2$  for the geometry of a single boundary at  $z = z_2$ . For the geometry of a single boundary at  $z = z_j$ , the FC in the region  $z < z_j$  is expressed as  $\langle \bar{\psi} \psi \rangle_j = \langle \bar{\psi} \psi \rangle_0 + \langle \bar{\psi} \psi \rangle_j^{(b)}$ , with the boundary-induced part being

$$\langle \bar{\psi} \psi \rangle_j^{(b)} = -\frac{2N(z/z_j)^{D+1}}{A_D a^D} \int_0^\infty dx x^D \frac{K_{\nu-1}(x)}{I_{\nu-1}(x)} I_\nu(xz/z_j) I_{\nu-1}(xz/z_j). \quad (3.17)$$

This expression is finite for points away from the boundary and vanishes at the AdS boundary. Similar to the case of the region to the right, the FC defined by (3.17) does not depend on the location of the boundary for a fixed distance from it. Note that the FC is not symmetric with respect to the boundary, the reason for this being that, though the background spacetime is homogeneous, the boundary  $z = z_j$  has a nonzero extrinsic curvature tensor and the two sides of the boundary are not equivalent. Here the situation is similar to that for curved boundaries in the Minkowski bulk.

Let us consider the asymptotic behavior of the single boundary-induced part. At large distances from the boundary,  $z \gg z_j$ , we introduce in (3.15) a new integration variable  $y = xz/z_1$  and expand the integrand by using the formulas for the modified Bessel functions for small values of the argument. To leading order, the remaining integral which involves the product of two MacDonald functions is evaluated by using a formula from [15]. In this way, one finds that

$$\langle \bar{\psi} \psi \rangle_j^{(b)} \approx -\frac{N a^{-D} (z_j/z)^{2\nu}}{2^{D+2\nu+1} \pi^{(D-1)/2}} \frac{D\Gamma(D/2 + 2\nu)\Gamma(D/2 + \nu)}{\nu\Gamma(D/2 + \nu + 1/2)\Gamma^2(\nu)}. \quad (3.18)$$

For the region to the left of the boundary, assuming  $z \ll z_j$ , by direct expansion of the integrand in (3.17), to leading order one gets

$$\langle \bar{\psi}\psi \rangle_j^{(b)} \approx -\frac{2^{2-2\nu} N(z/z_j)^{D+2\nu}}{A_D a^D \nu \Gamma^2(\nu)} \int_0^\infty dx x^{D+2\nu-1} \frac{K_{\nu-1}(x)}{I_{\nu-1}(x)}. \quad (3.19)$$

This expression gives the asymptotic behavior of the FC near the AdS boundary. Note that both limits, (3.18) and (3.19), correspond to distances from the boundary much larger than the curvature scale for the background spacetime:  $|y - y_j| \gg a$ . As we see, in these regions the boundary-induced part decays exponentially with the distance from the boundary.

For points near the boundary,  $|1 - z/z_j| \ll 1$ , the dominant contributions to the integrals in (3.15) and (3.17) come from large values of  $x$ . By using the asymptotic expressions for the modified Bessel functions for large values of the argument, to leading order we find

$$\langle \bar{\psi}\psi \rangle_j^{(b)} \approx -\frac{N\Gamma((D+1)/2)}{(4\pi)^{(D+1)/2} |y - y_j|^D}. \quad (3.20)$$

This leading term coincides with the corresponding one for the boundary in Minkowski spacetime.

### 3.2 FC in the region between the two boundaries

Now we return to the geometry with two boundaries. Using of the integration formulas (3.14) in (3.6), the FC in the region  $z_1 < z < z_2$  can be expressed in the form

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= \langle \bar{\psi}\psi \rangle_0 + \langle \bar{\psi}\psi \rangle_1^{(b)} + \frac{2N(z/z_1)^{D+1}}{A_D a^D} \int_0^\infty dx x^D \\ &\quad \times \Omega_\nu^{(1)}(x, \eta x) G_{\nu, \nu}(x, xz/z_1) G_{\nu, \nu-1}(x, xz/z_1), \end{aligned} \quad (3.21)$$

where  $\langle \bar{\psi}\psi \rangle_1^{(b)}$  is given by (3.15) with  $j = 1$ . For points outside the boundaries renormalization is only needed for the first term on the rhs. In formula (3.21), the functions  $\Omega_\nu^{(1)}(x, y)$  and  $G_{\nu, \nu-1}(x, y)$  are positive. Using the result that the function  $I_\nu(x)/K_\nu(x)$  is a monotonically increasing in terms of  $x > 0$ , we see that the function  $G_{\nu, \nu}(x, y)$  is negative for  $x < y$  and positive for  $x > y$ . Hence, the last term in (3.21) is negative. Combining this with the result  $\langle \bar{\psi}\psi \rangle_1^{(b)} < 0$ , we conclude that the boundary-induced part of the FC in the region between the boundaries is negative.

It can be checked that, in the region between the boundaries, the FC can also be written as

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= \langle \bar{\psi}\psi \rangle_0 + \langle \bar{\psi}\psi \rangle_2^{(b)} - \frac{2N(z/z_2)^{D+1}}{A_D a^D} \int_0^\infty dx x^D \\ &\quad \times \Omega_\nu^{(2)}(x/\eta, x) G_{\nu-1, \nu-1}(x, xz/z_2) G_{\nu-1, \nu}(x, xz/z_2), \end{aligned} \quad (3.22)$$

with

$$\Omega_\nu^{(2)}(x, y) = \frac{I_\nu(x)/I_{\nu-1}(y)}{K_\nu(x)I_{\nu-1}(y) + I_\nu(x)K_{\nu-1}(y)}. \quad (3.23)$$

In (3.22),  $\langle \bar{\psi}\psi \rangle_2^{(b)}$  is given by (3.17) with  $j = 2$  and the last term is induced by the boundary at  $z = z_1$ . The latter vanishes in the limit  $z_1 \rightarrow 0$ , it is finite for  $z = z_2$ , and diverges on the left boundary. Divergences occurring in this term are the same as those for a single boundary at  $z = z_2$ . All functions in the integrand of the last term in (3.22) are positive. From the discussion above it follows that, for the region between the boundaries, if we express the FC in the form

$$\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}\psi \rangle_0 + \sum_{j=1,2} \langle \bar{\psi}\psi \rangle_j^{(b)} + \Delta \langle \bar{\psi}\psi \rangle, \quad (3.24)$$

then the last (interference) term is finite everywhere, including the points on the boundaries. The surface divergences are contained in the single boundary parts. At large distances between the boundaries, as compared with the curvature radius of the background spacetime,  $(y_2 - y_1) \gg a$ , the interference part is exponentially suppressed.

Consider now the Minkowskian limit of the expression for the FC in the region between the boundaries. This corresponds to  $ma \gg 1$  for a fixed value of  $y$ , hence, one has  $\nu \gg 1$  and  $z/z_j \approx 1 + (y - y_j)/a$ . Introducing in the formulae above a new integration variable  $u = x/\nu$  and using the uniform asymptotic expansions for the modified Bessel functions for large values of the order, after some transformations, to leading order we get

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &\approx \langle \bar{\psi}\psi \rangle_{(M)} = -\frac{N}{A_D} \int_m^\infty dx \frac{(x^2 - m^2)^{D/2-1}}{\frac{x+m}{x-m} e^{2x(y_2-y_1)} + 1} \\ &\times \{ (m+x)[e^{2x(y-y_1)} + e^{2x(y_2-y)}] - 2m \}, \end{aligned} \quad (3.25)$$

where  $\langle \bar{\psi}\psi \rangle_{(M)}$  is the FC for boundaries in a Minkowski bulk. The expression (3.25) for  $\langle \bar{\psi}\psi \rangle_{(M)}$  is a special case of a more general formula derived in [20] for the Minkowski bulk with compact spatial dimensions. The fermionic condensate for a massless field has been considered in [21].

## 4 VEV of the energy-momentum tensor

The VEV of the energy-momentum tensor is another important local characteristic of the fermionic vacuum. In order to find this VEV we use the mode-sum formula

$$\langle T_{\mu\alpha} \rangle = \frac{i}{2} \int d\mathbf{k} \sum_{\sigma=1}^{N/2} \sum_{n=1}^{\infty} [\bar{\psi}_{\beta}^{(-)} \gamma_{(\mu} \nabla_{\alpha)} \psi_{\beta}^{(-)} - (\nabla_{(\mu} \bar{\psi}_{\beta}^{(-)}) \gamma_{\alpha)} \psi_{\beta}^{(-)}], \quad (4.1)$$

where the brackets denote symmetrization over the indices enclosed. Note that for the covariant derivative of the Dirac adjoint field one has  $\nabla_{\mu} \bar{\psi}_{\beta}^{(-)} = \partial_{\mu} \bar{\psi}_{\beta}^{(-)} - \bar{\psi}_{\beta}^{(-)} \Gamma_{\mu}$ . We see that the spin connection appears in the expression for the VEV in the form  $\gamma_{(\mu} \Gamma_{\nu)} + \Gamma_{(\mu} \gamma_{\nu)}$ . By using the expressions (2.5) it can be seen that this combination vanishes. Hence, in the evaluation of the VEVs we can make the replacement  $\nabla_{\nu} \rightarrow \partial_{\nu}$ .

Substituting (2.33) for the negative-energy mode functions into (4.1), it can be seen that the off-diagonal components vanish. For the VEVs of the diagonal components we find the following expressions (no summation over  $\mu$ )

$$\langle T_{\mu}^{\mu} \rangle = \frac{N\pi^2 a^{-D-1} (z/z_1)^{D+2}}{4(4\pi)^{(D-1)/2} \Gamma((D-1)/2)} \int_0^\infty du u^{D-2} \sum_{n=1}^{\infty} \frac{\gamma_{\nu,n} T_{\nu}(\eta, \gamma_{\nu,n})}{\sqrt{\gamma_{\nu,n}^2 + u^2}} f_{\nu}^{(\mu)}(\gamma_{\nu,n}, \gamma_{\nu,n} z/z_1), \quad (4.2)$$

with the notations

$$\begin{aligned} f_{\nu}^{(0)}(x, y) &= -(x^2 + u^2) [g_{\nu,\nu}^2(x, y) + g_{\nu,\nu-1}^2(x, y)], \\ f_{\nu}^{(l)}(x, y) &= \frac{u^2}{D-1} [g_{\nu,\nu}^2(x, y) + g_{\nu,\nu-1}^2(x, y)], \\ f_{\nu}^{(D)}(x, y) &= x^2 \left[ g_{\nu,\nu}^2(x, y) + g_{\nu,\nu-1}^2(x, y) - \frac{2\nu-1}{y} g_{\nu,\nu}(x, y) g_{\nu,\nu-1}(x, y) \right], \end{aligned} \quad (4.3)$$

and  $l = 1, \dots, D-1$ . As in the case of the FC, here we assume the presence of a cutoff function which makes the expressions (4.2) finite.

Further transformations of the VEVs proceeds similarly to those in the case of the FC. First, the series over  $n$  in (4.2) is transformed via the summation formula (3.4). Next, for the part corresponding

to the second term on the rhs of (3.4) we use the integration formula (3.14). In this way, the VEVs are expressed as (no summation over  $\mu$ )

$$\langle T_\mu^\mu \rangle = \langle T_\mu^\mu \rangle_1 + \frac{N(z/z_1)^{D+2}}{A_D a^{D+1}} \int_0^\infty dx x^{D+1} \Omega_\nu^{(1)}(x, \eta x) F_{1\nu}^{(\mu)}(x, xz/z_1), \quad (4.4)$$

where the notations

$$\begin{aligned} F_{1\nu}^{(l)}(x, y) &= \frac{1}{D} [G_{\nu, \nu}^2(x, y) - G_{\nu, \nu-1}^2(x, y)], \\ F_{1\nu}^{(D)}(x, y) &= G_{\nu, \nu-1}^2(x, y) - G_{\nu, \nu}^2(x, y) + \frac{2\nu-1}{y} G_{\nu, \nu}(x, y) G_{\nu, \nu-1}(x, y), \end{aligned} \quad (4.5)$$

are introduced with  $l = 0, \dots, D-1$ . In (4.4), the first term comes from the first integral in the summation formula (3.4) and it has the form (no summation over  $\mu$ )

$$\begin{aligned} \langle T_\mu^\mu \rangle_1 &= \frac{N a^{-D-1} (z/z_1)^{D+2}}{2(4\pi)^{(D-1)/2} \Gamma((D-1)/2)} \int_0^\infty du u^{D-2} \\ &\times \int_0^\infty dx \frac{x}{\sqrt{x^2 + u^2}} \frac{f_\nu^{(\mu)}(x, xz/z_1)}{J_\nu^2(x) + Y_\nu^2(x)}. \end{aligned} \quad (4.6)$$

This term corresponds to the VEV of the energy-momentum tensor in the region  $z > z_1$  for the geometry of a single boundary at  $z = z_1$ . The second term on the rhs of (4.4) is induced by the presence of the second boundary at  $z = z_2$ . The latter is finite for  $z_1 \leq z < z_2$  and renormalization is needed for the first term, only.

The single boundary part (4.6) could also be directly obtained by using the corresponding mode functions. The latter are given by (2.26) and (2.33), with  $0 \leq \lambda < 0$  and with the normalization coefficient (3.9).

The transformation of the part  $\langle T_\mu^\mu \rangle_1$  is also similar to that for corresponding term in the FC. Using (3.10) and the similar identity

$$\frac{g_{\nu, \mu}^2(x, y)}{J_\nu^2(x) + Y_\nu^2(x)} = J_\mu^2(y) - \frac{1}{2} \sum_{s=1,2} \frac{J_\nu(x)}{H_\nu^{(s)}(x)} H_\mu^{(s)2}(y), \quad (4.7)$$

the integrand in (4.6) can be decomposed into parts containing Bessel and Hankel functions, with the argument  $xz/z_1$ . Then, rotating the integration contour over  $x$  by an angle  $\pi/2$  ( $-\pi/2$ ), for the part with the function  $H_\mu^{(1)}(xz/z_1)$  ( $H_\mu^{(2)}(xz/z_1)$ ), we get (no summation over  $\mu$ )

$$\langle T_\mu^\mu \rangle_1 = \langle T_\mu^\mu \rangle_0 + \langle T_\mu^\mu \rangle_1^{(b)}, \quad (4.8)$$

where the two terms are respectively given by

$$\langle T_\mu^\mu \rangle_0 = \frac{(4\pi)^{-(D-1)/2} N}{2\Gamma((D-1)/2) a^{D+1}} \int_0^\infty dx x^{D-2} \int_0^\infty du \frac{u f_{0\nu}^{(\mu)}(u)}{\sqrt{u^2 + x^2}}, \quad (4.9)$$

and

$$\langle T_\mu^\mu \rangle_1^{(b)} = -\frac{N(z/z_1)^{D+2}}{A_D a^{D+1}} \int_0^\infty dx x^{D+1} \frac{I_\nu(x)}{K_\nu(x)} S_{1\nu}^{(\mu)}(xz/z_1). \quad (4.10)$$

In (4.9), the expressions for  $f_{0\nu}^{(\mu)}(y)$  are obtained from the corresponding expressions for  $f_\nu^{(\mu)}(x, y)$  by the replacement  $g_{\nu, \alpha}(x, y) \rightarrow J_\alpha(y)$ , and in (4.10) we have defined

$$\begin{aligned} S_{1\nu}^{(l)}(x) &= \frac{1}{D} [K_\nu^2(x) - K_{\nu-1}^2(x)], \\ S_{1\nu}^{(D)}(x) &= K_{\nu-1}^2(x) - K_\nu^2(x) + \frac{2\nu-1}{x} K_\nu(x) K_{\nu-1}(x), \end{aligned} \quad (4.11)$$

with  $l = 0, \dots, D-1$ . As we see, in both cases of single and double boundary geometries the energy density is equal to the stresses along the directions parallel to the boundaries. This property is related to the symmetry of the problem under consideration.

$\langle T_\mu^\mu \rangle_1^{(b)}$  vanishes in the limit  $z_1 \rightarrow 0$  (for the corresponding asymptotic behavior see below) while  $\langle T_\alpha^\mu \rangle_0$  can be identified with the VEV of the energy-momentum tensor in the boundary-free AdS bulk. With the decomposition (4.8) and for points away from the boundary, renormalization is required for this last part only. Because of the maximal symmetry of the background geometry, the renormalized VEV does not depend on the spacetime position and is completely determined by the trace:  $\langle T_\alpha^\mu \rangle_0 = \langle T_\sigma^\sigma \rangle_0 \delta_\alpha^\mu / (D+1)$ . For the case  $D = 3$  this VEV was investigated in [22] using the zeta-function techniques and also Pauli-Villars regularization. In what follows we specifically discuss the effects induced by the boundaries.

For a massless field one has  $\nu = 1/2$  and from (4.10) it can be easily seen that  $\langle T_\alpha^\mu \rangle_1^{(b)} = 0$  for  $z > z_1$ . We could obtain this result directly, by taking into account that for a massless fermionic field the problem is conformally related to the corresponding problem for a single boundary in the Minkowski bulk and for the latter geometry the VEV of the energy-momentum tensor vanishes. In the region between two boundaries, by making use of the expressions  $G_{1/2,1/2}(x, y) = (xy)^{-1/2} \sinh(x - y)$  and  $G_{1/2,-1/2}(x, y) = (xy)^{-1/2} \cosh(x - y)$ , from (4.4) we get (no summation over  $\mu$ )

$$\langle T_\mu^\mu \rangle = \langle T_\mu^\mu \rangle_0 + \left(\frac{z}{a}\right)^{D+1} \langle T_\mu^\mu \rangle_{(M),m=0}. \quad (4.12)$$

where

$$\langle T_\mu^\mu \rangle_{(M),m=0} = -\frac{N(1-2^{-D})\Gamma((D+1)/2)}{(4\pi)^{(D+1)/2}(z_2-z_1)^{D+1}} \zeta_R(D+1), \quad (4.13)$$

for  $\mu = 0, \dots, D-1$ , and  $\langle T_D^D \rangle_{(M),m=0} = -D\langle T_0^0 \rangle_{(M),m=0}$ . In (4.12),  $\zeta_R(x)$  is the Riemann zeta function. Here,  $\langle T_\alpha^\mu \rangle_{(M),m=0}$  is the corresponding VEV for two boundaries at  $z = z_1$  and  $z = z_2$  in Minkowski spacetime. Of course, (4.12) shows the standard conformal relation between the problems in AdS and Minkowski bulks. Note that for a massless field the boundary-free part  $\langle T_\alpha^\mu \rangle_0$  is completely determined by the trace anomaly.

We can write the VEV of the energy-momentum tensor in an alternative form (no summation over  $\mu$ ):

$$\langle T_\mu^\mu \rangle = \langle T_\mu^\mu \rangle_0 + \langle T_\mu^\mu \rangle_2^{(b)} + \frac{N(z/z_2)^{D+2}}{A_D a^{D+1}} \int_0^\infty dx x^{D+1} \Omega_\nu^{(2)}(x/\eta, x) F_{2\nu}^{(\mu)}(x, xz/z_2), \quad (4.14)$$

where the term

$$\langle T_\mu^\mu \rangle_2^{(b)} = -\frac{N(z/z_2)^{D+2}}{A_D a^{D+1}} \int_0^\infty dx x^{D+1} \frac{K_{\nu-1}(x)}{I_{\nu-1}(x)} S_{2\nu}^{(\mu)}(xz/z_2), \quad (4.15)$$

is the part in the VEV induced in the region  $z < z_2$  by a single boundary at  $z = z_2$  when the boundary at  $z = z_1$  is absent. In (4.14) and (4.15) we have introduced the notations

$$\begin{aligned} S_{2\nu}^{(l)}(x) &= \frac{1}{D} [I_{\nu-1}^2(x) - I_\nu^2(x)], \\ S_{2\nu}^{(D)}(x) &= I_\nu^2(x) - I_{\nu-1}^2(x) + \frac{2\nu-1}{x} I_\nu(x) I_{\nu-1}(x), \end{aligned} \quad (4.16)$$

and

$$\begin{aligned} F_{2\nu}^{(l)}(x, y) &= \frac{1}{D} [G_{\nu-1, \nu-1}^2(x, y) - G_{\nu-1, \nu}^2(x, y)], \\ F_{2\nu}^{(D)}(x, y) &= G_{\nu-1, \nu}^2(x, y) - G_{\nu-1, \nu-1}^2(x, y) - \frac{2\nu-1}{y} G_{\nu-1, \nu-1}(x, y) G_{\nu-1, \nu}(x, y), \end{aligned} \quad (4.17)$$

with  $l = 0, \dots, D-1$ . The last term in (4.14) is induced by the right boundary. It is finite for  $z_1 < z \leq z_2$  and diverges at  $z = z_1$ . It can be seen that, for a massless field one gets (no summation

over  $\mu$ )  $\langle T_\mu^\mu \rangle_2^{(b)} = (z/a)^{D+1} \langle T_\mu^\mu \rangle_{(M),m=0}$ , where  $\langle T_\mu^\mu \rangle_{(M),m=0}$  is given by (4.12) with  $z_1 = 0$ . As we see, in the region  $z < z_2$  the problem with a single boundary in the AdS bulk is conformally related to the problem in the Minkowski spacetime with two boundaries. This is a consequence of the boundary condition we have imposed on the AdS boundary.

It can be checked that both the single-boundary induced part,  $\langle T_\alpha^\mu \rangle_j^{(b)}$ , and the second boundary-induced part (last terms in (4.14) and (4.15)) in the VEV of the energy-momentum tensor obey the trace relation  $T_\mu^\mu = m\bar{\psi}\psi$ . In addition, the VEV obeys the covariant continuity equation  $T_{\alpha;\mu}^\mu = 0$  which, for the geometry under consideration, reduces to the single equation

$$z^{D+2} \partial_z (T_D^D / z^{D+1}) + T_\mu^\mu = 0. \quad (4.18)$$

Using the trace relation and taking into account that the boundary-induced part in the FC is negative everywhere, from (4.18) we conclude that the boundary-induced part in the VEV of the normal stress is a monotonically increasing function of  $z$  for all points outside the boundaries.

Taking into account that  $I_\nu(x) < I_{\nu-1}(x)$  and  $K_\nu(x) > K_{\nu-1}(x)$ , from (4.10) and (4.15) we see that, for the geometry of a single boundary at  $z = z_j$ , the boundary-induced part in the energy density is negative everywhere:  $\langle T_0^0 \rangle_j^{(b)} < 0$ . Using the properties of the modified Bessel functions, it can be seen that  $S_{j\nu}^{(D)}(x) > 0$  for  $z > z_j$  and  $S_{j\nu}^{(D)}(x) < 0$  for  $z < z_j$ . From here it follows that  $\langle T_D^D \rangle_j^{(b)} < 0$  in the region  $z > z_j$  and  $\langle T_D^D \rangle_j^{(b)} > 0$  in the region  $z < z_j$ . Next, it can be checked that

$$\begin{aligned} G_{\nu,\nu-1}(x,y) &> -G_{\nu,\nu}(x,y) > 0, \quad x < y, \\ G_{\nu-1,\nu}(x,y) &> G_{\nu-1,\nu-1}(x,y) > 0, \quad x > y, \end{aligned} \quad (4.19)$$

From these relations we see that  $F_{j\nu}^{(0)}(x,y) < 0$  and the parts of the energy density induced by the second plate (last terms in (4.4) and (4.14) with  $\mu = 0$ ) are negative. Hence, for the geometry of two boundaries the energy density is negative everywhere.

Now let us consider the asymptotics for the single boundary parts in the VEV of the energy-momentum tensor at small and large distances from the boundary. For the boundary at  $z = z_j$ , at large distances,  $z \gg z_j$ , to leading order one has

$$\langle T_D^D \rangle_j^{(b)} = \frac{D \langle T_0^0 \rangle_j^{(b)}}{D + 2\nu} = -D \frac{N m a^{-D} (z_j/z)^{2\nu} \Gamma(D/2 + \nu) \Gamma(D/2 + 2\nu)}{2^{D+2\nu+2} \pi^{(D-1)/2} \nu \Gamma^2(\nu) \Gamma(D/2 + \nu + 3/2)}, \quad (4.20)$$

with  $z_j/z = e^{-(y-y_j)/a}$ . For the region  $z < z_j$  with the condition  $z \ll z_j$ , the leading order terms have the form:

$$\langle T_D^D \rangle_j^{(b)} \approx -\frac{D}{2\nu} \langle T_0^0 \rangle_j^{(b)} \approx \frac{N a^{-D-1} (z/z_j)^{D+2\nu}}{2^{2\nu-1} A_D \nu \Gamma^2(\nu)} \int_0^\infty dx x^{D+2\nu-1} \frac{K_{\nu-1}(x)}{I_{\nu-1}(x)}. \quad (4.21)$$

The relations between the energy density and the normal stress, given by (4.20) and (4.21), can also be obtained by using the continuity equation (4.18) for  $\langle T_\alpha^\mu \rangle_j^{(b)}$  with  $\langle T_\mu^\mu \rangle_j^{(b)} = D \langle T_0^0 \rangle_j^{(b)} + \langle T_D^D \rangle_j^{(b)}$ . As it is seen from (4.20) and (4.21), at distances from the boundary larger than the AdS curvature scale, the boundary-induced part in the VEV of the energy-momentum tensor decays exponentially. For a scalar field with curvature coupling parameter  $\xi$  and with Robin boundary condition at  $z = z_j$ , for the VEV of the energy-momentum tensor one has [10]:  $\langle T_\mu^\mu \rangle_j^{(b)} \propto (z_j/z)^{2\nu_{sc}}$  for  $z \gg z_j$  and  $\langle T_\mu^\mu \rangle_j^{(b)} \propto (z_j/z)^{D+2\nu_{sc}}$  for  $z \ll z_j$ , where  $\nu_{sc} = \sqrt{D^2/4 - D(D+1)\xi + m^2 a^2}$ . In particular, for a conformally coupled scalar one has  $\nu_{sc} = \sqrt{m^2 a^2 + 1/2}$ , and the suppression of the VEVs is weaker than in the fermionic case with the same value of the nonzero mass.

For points near the boundary, the dominant contributions to the integrals in (4.10) and (4.15) come from large values of  $x$ . By using the asymptotic expressions for the modified Bessel functions for large values of the argument, we get the following leading behavior:

$$\langle T_0^0 \rangle_j^{(b)} \approx \frac{(D-1)a}{D(y-y_j)} \langle T_D^D \rangle_j^{(b)} \approx -\frac{N m \Gamma((D+1)/2)}{D(4\pi)^{(D+1)/2} |y-y_j|^D}. \quad (4.22)$$

For a boundary in the Minkowski spacetime the leading terms for the energy density and stresses along directions parallel to the boundary coincide with (4.22), whereas the normal stress vanishes.

The Minkowskian limit of the formulas for the VEV of the energy-momentum tensor is taken in a way similar to that we used for the case of FC. For  $ma \gg 1$  and for a fixed value of  $y$ , to leading order we find (no summation over  $\mu$ ):

$$\langle T_\mu^\mu \rangle \approx \langle T_\mu^\mu \rangle_{(M)} = -\frac{N}{A_D} \int_m^\infty dx \frac{(x^2 - m^2)^{D/2-1}}{\frac{x+m}{x-m} e^{2x(y_2-y_1)} + 1} G_{(M)}^{(\mu)}(x), \quad (4.23)$$

where

$$G_{(M)}^{(\mu)}(x) = \frac{x^2 - m^2}{D} \left\{ 2 + \frac{m}{x-m} \left[ e^{2x(y-y_1)} + e^{2x(y_2-y)} \right] \right\}, \quad (4.24)$$

for  $\mu = 0, 1, \dots, D-1$ , and  $G_{(M)}^{(D)}(x) = -2x^2$ . Here,  $\langle T_\mu^\mu \rangle_{(M)}$  is the VEV for the geometry of two boundaries in the Minkowski bulk. The formula (4.23) is a special case of the result given in [20]. The fermion Casimir energy for two parallel plates in 4-dimensional Minkowski spacetime has been investigated in [23] and [24] for massless and massive fields, respectively. The corresponding result for arbitrary number of dimensions is generalized in [25]. The topological Casimir effect and the VEV of the fermionic current for a massive fermionic field in a spacetime with an arbitrary number of toroidally compact dimensions have been considered in [26].

In Fig. 1, for the geometry of a single boundary, located at  $y = 0$ , we display the boundary-induced parts of the VEV of the energy density ( $\mu = 0$ , full curve) and of the normal stress ( $\mu = D$ , dashed curve), as functions of  $y/a$ . The latter measures the distance from the boundary in units of the AdS curvature radius. The graphs are plotted for a fermionic field in 4-dimensional AdS spacetime ( $D = 3$ ) and for the mass we have taken  $ma = 1$ .

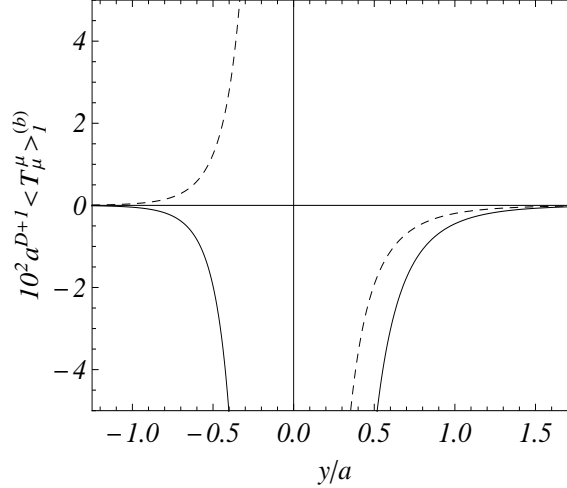


Figure 1: Boundary-induced parts of the VEV of the energy density (full curve) and of the normal stress (dashed curve), induced by a single boundary, at  $y = 0$ , as functions of  $y/a$ . Both plots are for  $D = 3$  and  $ma = 1$ .

## 5 Interaction forces and the Casimir energy

The force acting per unit surface of the boundary at  $z = z_j$  (vacuum effective pressure) can be obtained evaluating the normal stress at the location of the boundary:  $p^{(j)} = -\langle T_D^D \rangle_{z=z_j}$ . The boundary-free parts of the force acting from the left- and from the right-hand sides of the boundary compensate and

the resulting force is determined by the boundary induced part. By using the decomposition for the VEV of the energy-momentum tensor, in the region between the boundaries, the effective pressure is obtained as

$$p^{(j)} = p_1^{(j)} + p_{(\text{int})}^{(j)}, \quad (5.1)$$

where the first term on the rhs correspond to the situation when the second boundary is absent and the second term is induced by the presence of the second boundary. The latter can be termed as the interaction contribution. For the first part, one has  $p_1^{(j)} = -\langle T_D^D \rangle_{j,z=z_j}$ . In the regions  $z \leq z_1$  and  $z \geq z_2$ , the single boundary terms remain only:  $p^{(1)} = p_1^{(1)}$  for  $z < z_1$  and  $p^{(2)} = p_1^{(2)}$  for  $z > z_2$ . Because of the surface divergences in the single-boundary parts of the VEV, the term  $p_1^{(j)}$  is divergent and needs renormalization. The interaction part is finite for all nonzero values of the distance between the boundaries and it is not affected by the renormalization procedure.

The interaction parts of the effective pressure are obtained from the last terms in (4.4) and (4.14), by setting in the expressions for the  $D$ -components  $z = z_1$  and  $z = z_2$ , respectively. Using the Wronskian relation for the modified Bessel functions, one finds

$$p_{(\text{int})}^{(j)} = -\frac{N(z_j/z_1)^D}{A_D a^{D+1}} \int_0^\infty dx x^{D-1} \Omega_\nu^{(j)}(x, \eta x). \quad (5.2)$$

As we see, the corresponding effective pressures are always negative and, hence, the interaction forces are attractive. Using now the relations

$$\Omega_\nu^{(j)}(xz_1, xz_2) = (-1)^{j-1} z_j \partial_{z_j} \ln \left[ 1 + \frac{I_\nu(xz_1) K_{\nu-1}(xz_2)}{K_\nu(xz_1) I_{\nu-1}(xz_2)} \right], \quad (5.3)$$

the interaction parts can be expressed in a more convenient way as

$$p_{(\text{int})}^{(j)} = \frac{N(z_j/a)^{D+1}}{(-1)^j A_D} \partial_{z_j} \int_0^\infty dx x^{D-1} \ln \left[ 1 + \frac{I_\nu(xz_1) K_{\nu-1}(xz_2)}{K_\nu(xz_1) I_{\nu-1}(xz_2)} \right]. \quad (5.4)$$

Note that the forces acting on the left and on the right boundaries are different, in general. This property is related to the nonzero extrinsic curvature tensor for the the boundary geometry under consideration.

At small distances between the boundaries,  $\eta - 1 \ll 1$ , the dominant contribution comes from large values of  $x$  and, to leading order, we get

$$p_{(\text{int})}^{(j)} \approx -\frac{ND\Gamma((D+1)/2)}{(4\pi)^{(D+1)/2}(y_2 - y_1)^{D+1}}. \quad (5.5)$$

At large distances one has  $\eta \gg 1$ , and the leading terms in the asymptotic expansions are given by the expressions

$$\begin{aligned} p_{(\text{int})}^{(1)} &\approx -\frac{2^{2-2\nu} N a^{-D-1}}{A_D \Gamma^2(\nu) \eta^{D+2\nu}} \int_0^\infty dx x^{D+2\nu-1} \frac{K_{\nu-1}(x)}{I_{\nu-1}(x)}, \\ p_{(\text{int})}^{(2)} &\approx -\frac{2^{1-2\nu} N a^{-D-1}}{A_D \nu \Gamma^2(\nu) \eta^{2\nu}} \int_0^\infty dx \frac{x^{D+2\nu-1}}{I_{\nu-1}^2(x)}. \end{aligned} \quad (5.6)$$

Now, let us consider the Minkowskian limit corresponding to  $ma \gg 1$  for a fixed value of  $y$ . In this limit one has  $\eta \approx 1 + (y_2 - y_1)/a$ . Introducing in (5.2) a new integration variable,  $u = x/\nu$ , we use now the uniform asymptotic expansions for the modified Bessel functions for large values of the order. After some transformations, to leading order we have

$$p_{(\text{int})}^{(j)} \approx p_{(\text{M})} = -\frac{2N}{A_D} m^{D+1} \int_1^\infty dx \frac{x^2(x^2 - 1)^{D/2-1}}{\frac{x+1}{x-1} e^{2xm(y_2-y_1)} + 1}. \quad (5.7)$$



Of course, in the Minkowskian limit the forces are the same for the boundaries at  $y = y_1$  and  $y = y_2$ . Note that, for the geometry of a single boundary, in the Minkowskian limit the stresses on the left- and right-hand sides are the same by the symmetry of the problem. As a result, the corresponding net force vanishes and only the interaction part remains.

In Fig. 2, we depict the interaction forces for the model with  $D = 3$  as functions of the separation between boundaries as measured in units of the Compton wavelength of the fermionic particle. The dashed curve is the corresponding force for boundaries in Minkowski spacetime. The plots are for  $ma = 0.5$  (black),  $ma = 1$  (blue) and  $ma = 2$  (red). The curves on the left (right) of the dashed curve are for  $j = 1$  ( $j = 2$ ). As we see, with increasing  $a$  the forces tend to the corresponding result for the Minkowski bulk.

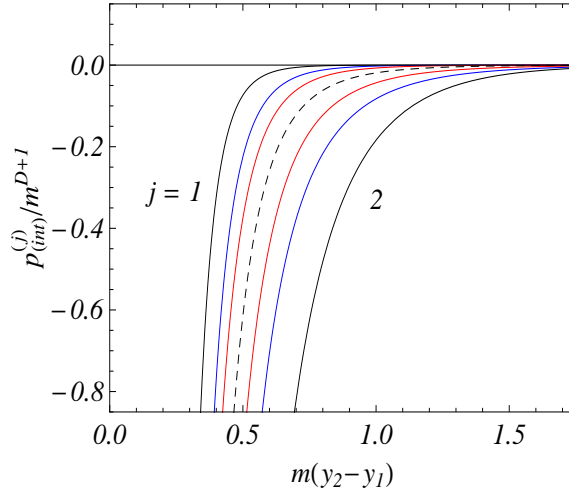


Figure 2: Interaction forces per unit surface as functions of the separation between the boundaries for the fermionic field in a 4-dimensional AdS bulk. The dashed curve corresponds to the force for boundaries in the Minkowski bulk. The plots correspond to  $ma = 0.5$  (black),  $ma = 1$  (blue) and  $ma = 2$  (red), respectively.

Now we consider the total vacuum energy in the region between the boundaries. The formal expression for this energy is obtained by integration of the energy density given by (4.2), with  $\mu = 0$ . Making use of standard formula for the integrals involving the square of a cylinder function, we get

$$\begin{aligned} E_{z_1 \leq z \leq z_2} &= \int_{z_1}^{z_2} dz \sqrt{|g|} \langle T_0^0 \rangle \\ &= -\frac{N}{2} \int \frac{d\mathbf{k}}{(2\pi)^{D-1}} \sum_{n=1}^{\infty} \sqrt{\gamma_{\nu,n}^2/z_1^2 + k^2}. \end{aligned} \quad (5.8)$$

This formula expresses the vacuum energy as a sum of ground state energies for elementary oscillators. Obviously, expression (5.8) is divergent and regularization, with the subsequent renormalization, is necessary. Here we follow the zeta function approach (for the application of the zeta function techniques to the calculations of the Casimir energy see [27] and references therein). Instead of (5.8), we consider the related zeta function

$$\zeta(s) = -\mu_0^{s+1} \frac{N}{2} \int \frac{d\mathbf{k}}{(2\pi)^{D-1}} \sum_{n=1}^{\infty} (\gamma_{\nu,n}^2/z_1^2 + k^2)^{-s/2}, \quad (5.9)$$

where the constant  $\mu_0$  with dimensions of mass is introduced by dimensional reasons (it is the regularization parameter). The expression on the rhs of (5.9) is finite for  $\text{Re } s > D$  and for the evaluation of the vacuum energy we need its analytic continuation at  $s = -1$ :  $E_{z_1 \leq z \leq z_2} = \zeta(s)|_{s=-1}$ .

Evaluating the integral over  $\mathbf{k}$ , the zeta function can be expressed in the form

$$\zeta(s) = -\frac{N\Gamma((s+1-D)/2)\mu_0^{s+1}}{2(4\pi)^{(D-1)/2}\Gamma(s/2)z_1^{D-1-s}}\zeta_1(s+1-D), \quad (5.10)$$

where we have introduced the partial zeta function

$$\zeta_1(s) = \sum_{n=1}^{\infty} \gamma_{\nu,n}^{-s}, \quad (5.11)$$

directly related to the eigenvalues  $\gamma_{\nu,n}$ . We need the analytic continuation of the function (5.11) to a neighborhood of  $s = -D$ . This procedure is standard in the theory of the Casimir effect and we only give the main steps.

From Cauchy's residue formula, the integral representation follows

$$\zeta_1(s) = \int_C \frac{du}{2\pi i} u^{-s} \partial_u \ln [ug_{\nu,\nu-1}(u, u\eta)], \quad (5.12)$$

where  $C$  is a closed, counterclockwise contour on the complex  $z$  plane enclosing all zeros  $\gamma_{\nu,n}$ . We take the contour made of a large semicircle (with radius tending to infinity) centered at the origin and placed to its right, plus a straight part overlapping the imaginary axis and avoiding the origin by a small semicircle  $C_\rho$  on the right half-plane with radius  $\rho$ . For small  $\rho$ , one has

$$\int_{C_\rho} \frac{du}{2\pi i} u^{-s} \partial_u \ln [ug_{\nu,\nu-1}(u, u\eta)] = B \frac{\rho^{2-s}}{2-s} \sin(\pi s/2), \quad (5.13)$$

being  $B$  a constant independent of  $s$ . Denoting the upper and lower halves of the contour  $C$  by  $C_1$  and  $C_2$ , respectively, the integral can be cast in the form

$$\begin{aligned} \zeta_1(s) &= \int_C \frac{dz}{2\pi i} u^{-s} \partial_u \ln [u^{1-\nu} J_{\nu-1}(u\eta)] \\ &+ \sum_{l=1,2} \int_{C_l} \frac{du}{2\pi i} u^{-s} \partial_u \ln [u^\nu H_\nu^{(\alpha)}(u)] \\ &+ \sum_{l=1,2} \int_{C_l} \frac{du}{2\pi i} u^{-s} \partial_u \ln \left[ 1 - \frac{J_\nu(u) H_{\nu-1}^{(\alpha)}(u\eta)}{H_\nu^{(\alpha)}(u) J_{\nu-1}(u\eta z)} \right]. \end{aligned} \quad (5.14)$$

After parameterizing the integrals over the imaginary axis and substituting into (5.10), we arrive at the following expression for the zeta function

$$\begin{aligned} \zeta(s) &= -\frac{(4\pi)^{(1-D)/2} N \mu_0^{s+1} z_1^{1+s-D}}{2\Gamma(s/2)\Gamma((D+1-s)/2)} \left\{ \frac{\pi B \rho^{D+1-s}}{D+1-s} \right. \\ &+ \int_\rho^\infty dx x^{D-1-s} \partial_x \left[ \ln (x^{1-\nu} I_{\nu-1}(x\eta)) \right. \\ &\left. \left. + \ln (x^\nu K_\nu(x)) + \ln \left( 1 + \frac{I_\nu(x) K_{\nu-1}(x\eta)}{K(x) I_{\nu-1}(x\eta)} \right) \right] \right\}, \end{aligned} \quad (5.15)$$

where we have used that  $\Gamma(y) \sin(\pi y) = \pi/\Gamma(1-y)$ . The term with the factor  $B$  vanishes in the limit  $\rho \rightarrow 0$  at the physical point  $s = -1$ , while the last term is finite at this point. The first (second) term in the square brackets in (5.15) corresponds to the vacuum energy in the region  $z \leq z_2$  ( $z \geq z_1$ ) for the geometry of a single boundary at  $z = z_2$  ( $z = z_1$ ) and will be denoted as  $E_{1,z \leq z_2}^{(2)}$  ( $E_{1,z \geq z_1}^{(1)}$ ). Adding

also the vacuum energies from the regions  $z \leq z_1$  and  $z \geq z_2$  (denoted as  $E_{1,z \leq z_1}^{(1)}$  and  $E_{1,z \geq z_2}^{(2)}$ ), for the total vacuum energy we finally get

$$E = E_{1,z \leq z_1}^{(1)} + E_{1,z \geq z_2}^{(2)} + E_{z_1 \leq z \leq z_2} = \sum_{j=1,2} E_1^{(j)} + \Delta E. \quad (5.16)$$

Here,  $E_1^{(j)}$  is the energy for the geometry of a single boundary at  $z = z_j$ , and the interference part  $\Delta E$  is given by the last term in (5.15) with  $s = -1$ . After integration by parts, one gets for the latter

$$\Delta E = -\frac{N}{A_D} \int_0^\infty dx x^{D-1} \ln \left[ 1 + \frac{I_\nu(xz_1)K_{\nu-1}(xz_2)}{K(xz_1)I_{\nu-1}(xz_2)} \right]. \quad (5.17)$$

The renormalization procedure to be carried out for the divergences of the single boundary parts in (5.16) is similar to those previously discussed within the framework of braneworld scenarios (see, e.g., [7, 8]). Note that the dependence of  $E_1^{(j)}$  on  $z_j$  is in the form  $z_j^{-D}$  and this dictates the form of the counterterms located on the boundaries. By finite renormalizations, the single boundary terms in (5.16) can be absorbed into the counterterms. The renormalized vacuum energy has the form  $E = \sum_{j=1,2} c_j z_j^{-D} + \Delta E$ , with renormalized coefficients  $c_j$ . Comparing (5.17) with (5.4), we find the relation

$$p_{(\text{int})}^{(j)} = -(-1)^j (z_j/a)^{D+1} \partial_{z_j} \Delta E, \quad (5.18)$$

between the interaction forces and the interference part of the Casimir energy.

Here we have considered the fermionic Casimir effect with bag boundary conditions on the background of AdS spacetime. In a similar way the fermionic Casimir densities can be evaluated for the Randall-Sundrum braneworld model. This model is formulated on 5-dimensional AdS spacetime, thus with a single extra dimension. The fifth dimension  $y$  is compactified on an orbifold  $S^1/Z_2$  of length  $L$ , with  $-L \leq y \leq L$ . The corresponding line element has the form (2.1) with the warp factor  $e^{-2|y|/a}$ . Two 3-branes are located at the orbifold fixed points,  $y = 0$  and  $y = L$ . In terms of the conformal radial coordinate  $z$ , for the branes one has  $z = a$  and  $z = z_L = ae^{L/a}$ . In the Randall-Sundrum model, depending on the parity of the spinor field under a chiral transformation, two types of boundary conditions arise on the branes. For these boundary conditions the eigenvalues of the quantum number  $\lambda$  are roots of the equation

$$g_{\nu-s, \nu-s}(\lambda a, \lambda z_L) = 0, \quad (5.19)$$

with  $s = 0, 1$  for even and odd fields, respectively. The summation formula for series over these eigenvalues is obtained from the general results of [18, 19]. Calculations are actually the same as those we have described above for the case of bag boundary conditions. In the normalization condition for the mode functions the integration over  $y$  goes over the region  $(-L, L)$ . As a result the normalization coefficient will have an additional factor  $1/2$ , as compared with the case where the problem is formulated on the interval  $(0, L)$ .

The FC and the VEV of the energy-momentum tensor for a fermionic field in the Randall-Sundrum model are obtained from the formulas given in Sects. 3 and 4 by changing the order of the appropriately modified Bessel function from  $\nu - 1$  to  $\nu$  for even fields and from  $\nu$  to  $\nu - 1$  for odd fields and by adding an extra factor of  $1/2$ . For example, in the case of even fields, to Eq. (3.15) we need only add the factor  $1/2$ , whereas to Eq. (3.17) we have to add the factor  $1/2$  and make also the replacements  $K_{\nu-1}(x) \rightarrow -K_\nu(x)$  and  $I_{\nu-1}(x) \rightarrow I_\nu(x)$ . In the case of odd fields the situation is just opposite: in (3.15) we add the factor  $1/2$  and replace  $K_\nu(x) \rightarrow -K_{\nu-1}(x)$  and  $I_\nu(x) \rightarrow I_{\nu-1}(x)$ , while in (3.17) we only add the factor  $1/2$ . Note that when evaluating the vacuum energy in the braneworld model the integration goes over the region  $(-L, L)$  and there is no need to add the factor  $1/2$  in the corresponding expressions.

Through the above mentioned replacements of the modified Bessel functions, from (5.17) we readily obtain the corresponding vacuum energies for even and odd fields in the Randall-Sundrum model. For

$D = 4$  the corresponding formulas were obtained in [8] (note that in this reference the effective Lagrangian is considered per fermionic degree of freedom, which corresponds to  $-\Delta E/N$ ). The VEV of the energy-momentum tensor for a bulk Dirac spinor in the Randall-Sundrum model has been considered in [11]. In this reference, for the case of a massive field, a general formula is given for the unrenormalized VEV only. To compare, in our approach, based on the generalized Abel-Plana formula, the pure AdS parts in the VEVs are extracted explicitly and, for the points away from the branes, the renormalization procedure is the same as for the boundary-free parts. In addition, the boundary induced parts are presented in terms of exponentially convergent integrals, which are very well suited for numerical calculations.

## 6 Conclusions

In this paper we have investigated the fermionic condensate and the VEV of the energy-momentum tensor for a massive fermionic field in AdS spacetime in the presence of two boundaries on which the field obeys bag boundary conditions. For the evaluation of the VEVs we have employed the mode summation technique. In the region between the boundaries, a complete set of positive- and negative-energy mode functions is given by (2.26) and (2.33), respectively, where the eigenvalues of the radial quantum number  $\lambda$  are determined from the boundary conditions and they are solutions of the equation (2.27). The mode sums for the FC and the energy-momentum tensor contain series over these eigenvalues. For the summation of the series we have used the generalized Abel-Plana formula (3.4), which allowed to separate the VEVs into single boundary and second boundary-induced parts. In this representation, explicit knowledge of the eigenvalues of  $\lambda$  is not necessary. The VEVs for the geometry of a single boundary are further decomposed into boundary-free and boundary-induced parts. As a result, in the region between the boundaries, the VEVs can be expressed in two equivalent ways, respectively given by Eqs. (3.21) and (3.22), for the FC, and by Eqs. (4.4) and (4.14), for the energy-momentum tensor. With these representations, and for points away from the boundaries, the boundary induced part is finite and renormalization is required for the boundary-free part, only.

For the geometry of a single boundary located at  $z = z_j$ , the boundary-induced contribution to the FC is given by (3.15), in the region  $z > z_j$ , and by (3.17), in the region  $z < z_j$ . This contribution is negative for both regions and it is not symmetric with respect to the boundary. Such fact is related to the existence of a nonzero extrinsic curvature tensor for the boundary  $z = z_j$  in the AdS bulk. At large distances from the boundary, as compared with the value of the AdS curvature radius, the boundary-induced parts are exponentially suppressed by the factors  $e^{-2\nu(y-y_j)/a}$ , in the region  $y > y_j$ , and  $e^{-(D+2\nu)(y-y_j)/a}$ , in the region  $y < y_j$ . In particular, the boundary-induced part vanishes on the AdS boundary. For points near the boundary at  $y = y_j$ , the leading term in the asymptotic expansion of the FC is given by (3.20) and it does coincide with the corresponding expression for the boundary in Minkowski spacetime. For the geometry of two boundaries, the FC in the region between them can be expressed in the form (3.24), where the interference term is finite everywhere, including the points on the boundaries. For large separation of the boundaries, as compared with the curvature radius of the background spacetime, the interference part is exponentially suppressed. The boundary-induced part of the FC in the region between the two boundaries is negative.

The boundary-induced contributions in the VEV of the energy-momentum tensor for a single boundary are given by (4.10) and (4.15), for the regions on the right and on the left of the boundary, respectively. The corresponding vacuum energy is negative, whereas the normal stress is negative on the right domain and positive on the left one. At large distances from a single boundary located at  $y = y_j$ , the boundary-induced terms decay as  $e^{-2\nu(y-y_j)/a}$ , for  $y > y_j$ , and as  $e^{-(D+2\nu)(y-y_j)/a}$ , for  $y < y_j$ . For points near the boundary the corresponding asymptotic behavior is given by (4.22).

The forces acting on the boundaries and the Casimir energy were considered in Sect. 5. The force acting on the boundary at  $z = z_j$  is decomposed as (5.1) where the first term on the rhs is the force for a single boundary (when the second one is absent) while the second term is induced

by the presence of the other boundary. The interaction part of the force is attractive and it can be expressed in the form (5.2) or, equivalently, as in (5.4). The forces acting on the left and on the right boundaries are different from each other. For small separations of the boundaries, as compared with the AdS curvature radius, to leading order we recover the result for the Minkowski bulk with boundaries. For large separations, the asymptotic expressions for the force are given by (5.6) and the interaction force is again exponentially suppressed. We have also checked with care the transition to the Minkowskian limit corresponding to  $a \rightarrow \infty$ . For the evaluation of the total vacuum energy in the region between the boundaries we have employed zeta function techniques. As the corresponding scheme is well described in the literature on the Casimir effect, we have here sketched the main steps only. For the total vacuum energy, including the contributions coming from the regions  $z \leq z_1$  and  $z \geq z_2$ , one has (5.16) with the interference part being given by (5.17). In the second part of Sect. 5 we have described in detail how from our results the corresponding formulae for fermionic Casimir densities in Randall-Sundrum-type braneworld scenarios immediately follow.

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